The Use of Hyperbolic Cosine Function in Catenary Bridge Structure

Research Question: The analysis of the relationship between force and other variables in different bridge supporting scenario.

Subject: Mathematics

Word count: 4514

CONTENTS

Introduction

I am a native from Suzhou, which an ancient city full of bridges. With numerous narrow rivers covering all over the city, Suzhou has a fame as the "Oriental Venice". In ancient time, to connect different parts in cities and facilitate travel for people, plenty of bridges were built along the river as *Figure1* below shows. Old bridges of Suzhou have a few properties. First, the number of them is grand. According to some statistics, the average number of bridges in Suzhou is 15 per square kilometer and it is the greatest number of bridges that a city has in the world. This "Oriental Venice" has even greater number of bridges than the world-famous city of Venice, Italy, which has the density of bridges that one bridge per square kilometer. Secondly, the timber and structure of bridges in Suzhou have obvious change along with time. The main materials are wood, slab stone, granite, limestone, etc. Third property is that each bridge has their own characteristics, that none of two bridges looks the same. The last peculiarity of bridges in Suzhou is that they are they carry the history and culture of Suzhou. By looking at a bridge, people is not learning about this bridge, but the structure, the raw material, the design and other appearance, but enjoying the profoundness of culture and the story behind the construction of the bridges.

My grandparents used to say, the best place to live in a house with a small bridge over the flowing stream aside and that was where I live with them before five years old. This gave me a lot of chances to observe bridges around. The most attractive thing that bridges brought to me is not their delicate appearance, but their strict construction.

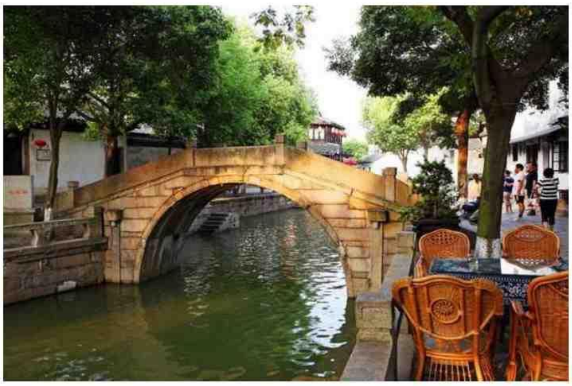

Figure1. A typical bridge in Suzhou.

The material, construction and design in ancient and the modern city are very different. Not until the 18th century, most of the bridges are constructed with wood, stone and iron and as the building materials. For example, the most famous Zhaozhou Bridge in Hebei province in china is the largest span stone arch bridge among all the ancient bridges and the first arch bridge open shoulder in the history of the world. Another famous stone bridge is the Lugou Bridge in Beijing. It was built in 12th century and famous around the world by 13th. It was built for the transportation that time and have large maximum applied forces. By observing the structure of these two bridges shows in *Figure2 & 3*, I found out that they all have similar structure of arch construction. According to the research result, these arches are used to given deck supports. The greater the number of apertures, the more stable the bridge is.

Figure2. The Zhaozhou bridge in Hebei.

Figure3. The Lugou Bridge in Beijing.

Moreover, there are other types of construction of bridges, such as cable stayed bridge, suspension bridge, frame bridge, beam bridge, etc.

Background

Nowadays, the most popular bridge style is cable stayed and suspension bridges. On a suspension bridge, there are three important components, tower, large cables hanging among tall tower and thinner cables hanging parallel from the large cables to the bridge deck to support the weight of the bridge as *Figure4(e)* shows. In this case, the towers will carry most of the weight and this make the bridge deck more rigidity. However, on a cable styed bridge, the cables run directly from the tall tower to the bridge deck and create the fan-like pattern as *Figure4(d)* shows. However, the more popular one is cable stayed designs since they take less time to build than suspension type. For the same length of the bridge, the cable stayed bridges require fewer cables than suspension one and anchorages are not required for them. Thus, the construction time and expenditure are significantly less than suspension bridges. Considering from the perspective of the stability of the structure, cable stayed bridge has both advantages and disadvantages. Comparing with suspension bridges, they are more rigidity since the cable on it can handle more pressure and the deformation of the duck within operational life will decrease. However, consider from another viewpoint, the cable stayed design do not have such strong flexibility as suspension bridges do. It is easier to be influenced by wind. To have a further investigation of the reason for high carrying capacity of cables on the cable stayed bridges, the content of catenary is explained following.

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Figure4. Different type of bridges.

Figure5. Catenary shape in cable stayed bridge.

Introduction of Catenary (Hyperbolic Cosine Function)

In the field of physics and geometry, the catenary is a shape that idealized hanging chain or cable that hangs under its own weight when supported only at its ends. it typically has a comparatively large distance from the horizontal positions where it's attached to the lowest point on the cable, measurement ***y*** as *Diagram1* shows. In another word, **y** is very large relative to the span length ***x***. Also, distance **y** is called a sag.

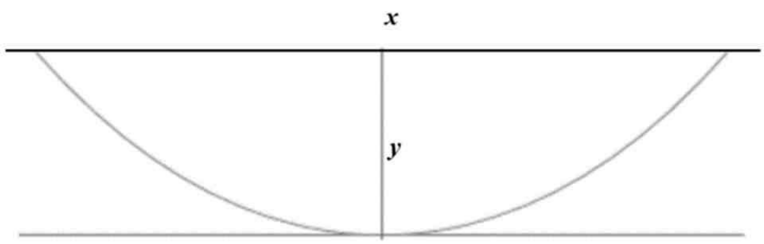


Diagram1. A catenary structure.

Mathematically, the shape of catenary can be described by the hyperbolic cosine function shows below:

$$\cosh y = \frac{e^{y} + e^{-y}}{2}$$

Also, the hyperbolic sine function is:

$$\sinh y = \frac{e^{y} - e^{-y}}{2} \qquad (1)$$

Before getting to know the function of a catenary (hyperbolic cosine function), the following calculus us going to deduce the inverse hyperbolic sine function.

Firstly, assume that:

$$x = \sinh y \qquad (2)$$

$$y = sinh^{-1}x \qquad (3)$$

Put (2) into (1):

$$x = \frac{e^{y} - e^{-y}}{2}$$

$$2x = e^{y} - e^{-y}$$

Multiply both sides by e^{y}:

$$e^y(2\text{x}) = e^y(e^y - e^{-y})$$

$$2\text{x} \cdot e^y = e^{2y} - 1$$

$$0 = e^{2y} - 2\text{x} \cdot e^y - 1$$

Consider e^y as the variable of the equation and solve this quadradic equation according to the Formula of roots shows below:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$e^y = \frac{2x \pm \sqrt{(2x)^2 - 4(1)(-1)}}{2}$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 + 1}}{2}$$

$$= x \pm \sqrt{x^2 + 1}$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

Take natural log on both sides:

$$\ln(e^y) = \ln(x \pm \sqrt{x^2 + 1})$$

$$y = \ln(x \pm \sqrt{x^2 + 1}) \quad (4)$$

According to (3), $\text{y} = sinh^{-1}x$.

Therefore, the inverse hyperbolic sine function is deduced.

$$sinh^{-1}x = \ln(x \pm \sqrt{x^2 + 1}) \quad (5)$$

Investigation of the Force on Catenary

The table below indicates all the measurement and variables in the following calculus according to the full analyzed *Diagram2* of a piece of cable.

Symbol	Explanation
x	Distance between two pivots.
s	length of the cable.
y	Distance from the origin to the highest point of the cable.
ω	weight per unit length.
W	Weight of the cable.
c	Distance from the origin to the lowest point of the cable
T_0	Tension on the horizontal direction.
T	Tension close to the support point on cable.
θ	Angle between T and horizontal axis.
dx	Horizontal side of zooming triangle.
dy	Vertical side of zooming triangle.
ds	Side of zooming triangle along the cable.

Table1. Variables in force analysis of catenary.

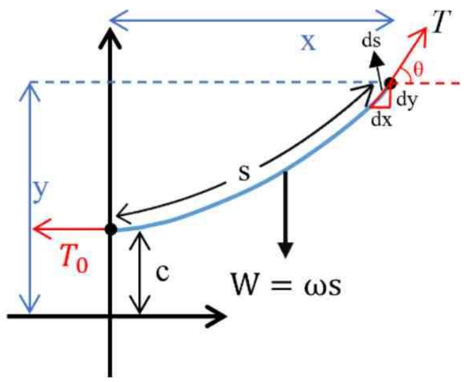


Diagram2. Detail of arc forces in the coordinate.

As the *Diagram3* shows, there are three cables that have different extents of catenary and the following calculus can be calculated. The string here is very light.

(a) Both pivots are on the vertically straight. The only load is the weight of the cable itself. The load on the cable is uniform and the force caused by gravity can calculated based on equation below:

$$W = \omega x$$

(b) There is a slight sag of the cable here and the length of the cable is slightly longer than the distance between two pivots. However, the length and the distance are close enough and force is still equally distributed. Therefore, the weight is the same as type (a).

$$W = \omega x$$

(c) There are many sags in this type and the forces (per unit length) is becoming closer in the x-direction, but the distances along the arc are the same. As a result, the total weight of the cable is now on the right side of the center of the cable and the specific location is unknown.

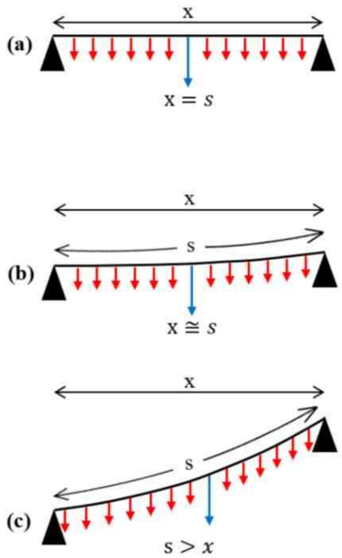


Diagram3. Three types of catenary.

Putting *Diagram3. (c)* into a coordinate axis as *Diagram4* shows. The y-axis passes through the lowest point on the cable and here is a distance (constant c) between the origin and the lowest point. Point D(xy) is a point on the cable.

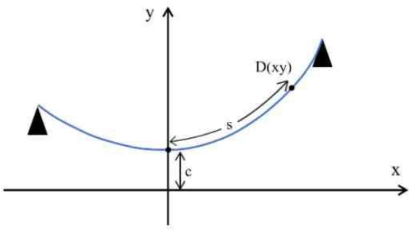


Diagram4. Catenary cable in coordinate

Zooming in *Diagram4*, the force analysis diagram contains all the forces act on this particular arc is shown in *Diagram5*. At the lowest point of the cable, the only tension act on it is the tension on the horizontal direction (T_0), which is the lowest on the cable. The total weight of the cable equals to the tension per unit length times the length of the cable along the arc. However, this is unknown.

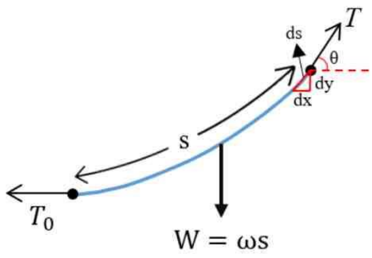


Diagram5. Enlarged arc between y-intercept and D.

The total three forces act on this arc can be combined into a triangle as *Diagram6* shows. Within this triangle, a formula can be calculated according to the Pythagorean theorem.

$$T = \sqrt{T_0^2 + \omega s^2}$$

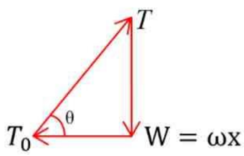


Diagram6. Triangle of forces.

Putting the partial arc back into the coordinate, the specific distance c is shown in *Diagram7*. Since distance c is a constant and it is unknown, the following calculus is processed to find the relationship among tensions and constant c.

$$T = \sqrt{T_0^2 + \omega s^2}$$

$$T = \omega\sqrt{\frac{T_0^2}{\omega^2} + s^2}$$

$$\text{Let } c = \frac{T_0}{\omega} \qquad (6)$$

$$T = \omega\sqrt{c^2 + s^2} \qquad (7)$$

$$T = \omega c\sqrt{1 + \frac{s^2}{c^2}} \qquad (8)$$

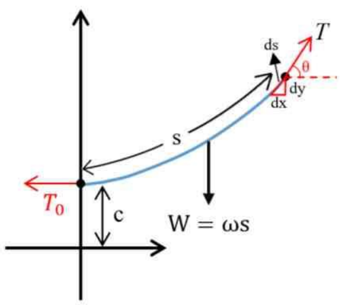


Diagram7. Arc forces in the coordinate.

According to (6), following can be calculated out:

$$\omega c = T_0 \qquad (9)$$

Therefore, the relationship between constant c and tension T_0 is founded.

Put (9) into (8):

$$T = T_0\sqrt{1 + \frac{s^2}{c^2}} \qquad (10)$$

At the end of the cable, there is another tension close to the support point and it points at a particular angle θ. Zooming a point on the cable shows in *Diagram5* and *Diagram7* and the three sides are ds, dx and dy relatively and they can be shown in the formula below:

$$\mathrm{ds} = \sqrt{dx^2 + dy^2}$$

Also, according to the circular functions, dx can be expressed as below shows:

$$\mathrm{dx} = \mathrm{ds}\cos\theta$$

$$\mathrm{dx} = \mathrm{ds}\frac{T_0}{T}$$

Substitute the T here by (10):

$$\mathrm{dx} = \mathrm{ds}\frac{T_0}{T_0\sqrt{1+\frac{s^2}{c^2}}}$$

$$\mathrm{dx} = \frac{ds}{\sqrt{1+\frac{s^2}{c^2}}} \qquad (11)$$

Integral both sides of (11):

$$\int_0^X \mathrm{dx} = \int_0^S \frac{ds}{\sqrt{1+\frac{s^2}{c^2}}}$$

$$\mathrm{x}|_0^X = c\left[sinh^{-1}(\frac{s}{c})\right]_0^S \qquad (12)$$

$$\mathrm{x} = \mathrm{c}\left[sinh^{-1}\left(\frac{s}{c}\right) - 0\right] \qquad (13)$$

<u>Premovement for Step (12)</u>

The inverse hyperbolic sine of **s**, which is proved in *Introduction* is shown below. Where **c** is a constant.

$$sinh^{-1}s = \ln(s + \sqrt{s^2 + c^2}) \qquad (5)$$

The following equation is going to be integrated.

$$\int \frac{1}{\sqrt{s^2+c^2}} ds \qquad (14)$$

$let\ s = c\tan\theta$

$ds = d\theta \cdot c\sec^2\theta$

$$= \int \frac{1}{\sqrt{(c\tan\theta)^2 + c^2}} ds$$

$$= \int \frac{1}{\sqrt{c^2(\tan^2\theta + 1)}} ds$$

$$= \int \frac{1}{c \sec \theta} ds \qquad (15)$$

Bring $ds = d\theta \cdot c \sec^2 \theta$ into (15)

$$= \int \frac{1}{c \sec \theta} d\theta \cdot c \sec^2 \theta$$

$$= \int sec\theta \cdot d\theta$$

Expend the function by $\tan \theta + \sec \theta$:

$$= \int \frac{\sec \theta \tan \theta + \sec^2 \theta}{\tan \theta + \sec \theta} d\theta$$

$let\ u = \tan \theta + \sec \theta$

$$d\theta = \frac{1}{\sec \theta \tan \theta + \sec^2 \theta} du$$

$$= \int \frac{\sec \theta \tan \theta + \sec^2 \theta}{u} \cdot du \cdot \frac{1}{\sec \theta \tan \theta + \sec^2 \theta}$$

$$= \int \frac{1}{\mathrm{u}} du$$

$$= \ln(u)$$

$$= \ln(\tan \theta + \sec \theta)$$

Let $\theta = \arctan(\mathrm{w})$

Since $\tan(\arctan(w)) = w$

$\sec(\arctan(w)) = \sqrt{w^2 + 1}$

$$= \ln(\sqrt{w^2 + 1} + w)$$

Let $w = \frac{s}{c}$

$$= \ln\left(\sqrt{\frac{s^2}{c^2} + 1} + \frac{s}{c}\right)$$

According to the solution of (14) shows above and (5):

$$sinh^{-1}\left(\frac{s}{c}\right) = \ln\left(\sqrt{\frac{s^2}{c^2} + 1} + \frac{s}{c}\right) = \int \frac{1}{\sqrt{s^2 + c^2}} ds$$

Therefore,

$$\int_0^S \frac{ds}{\sqrt{1 + \frac{s^2}{c^2}}} = c\left[sinh^{-1}(\frac{s}{c})\right]_0^s$$

As a result, step (12) is proved.

Now, returning to the followed calculus based on (13).

Since $\sinh(0) = 0 \rightarrow sinh^{-1}(0) = 0$, (13) can be written in:

$$x = c\, sinh^{-1}\left(\frac{s}{c}\right)$$

$$\frac{x}{c} = sinh^{-1}\left(\frac{s}{c}\right)$$

$$\frac{s}{c} = sinh\left(\frac{x}{c}\right)$$

$$s = c\, sinh\left(\frac{x}{c}\right) \qquad (16)$$

To conclude the previous calculus, three equations related to the relationship involves in constant c, the distance between the origin and the tach point in horizontal direction (x) and the distance between these two points in vertical direction (y) shows in *Diagram8*.

$$\boldsymbol{\omega c = T_0} \qquad \textbf{(9)}$$

$$\mathbf{T = \omega\sqrt{c^2 + s^2}} \qquad \textbf{(7)}$$

$$\mathbf{s = c\, sinh\frac{x}{c}} \qquad \textbf{(16)}$$

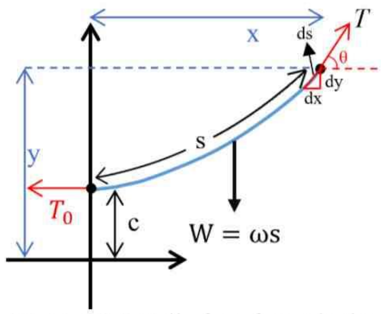


Diagram8. Detail of arc forces in the coordinate.

According to the previous triangle of three forces act on the cable, the following derivation can be completed:

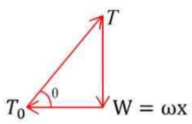


Diagram9. Triangle of forces.

$$\tan\theta = \frac{dy}{dx}$$

$$dy = (\tan\theta)dx$$

$$dy = \frac{W}{T_0}dx = \frac{\omega s}{\omega c}dx = \frac{s}{c}dx = \frac{c\, sinh\left(\frac{x}{c}\right)}{c}dx$$

$$dy = sinh\left(\frac{x}{c}\right)dx \qquad (17)$$

Integral both sides of (17):

$$\int_c^y dy = \int_0^x sinh\left(\frac{x}{c}\right)dx$$

$$y|_c^y = c\left[\cosh\frac{x}{c} - \cos(0)\right]$$

$$y - c = c\left[\cosh\frac{x}{c} - 1\right]$$

$$y - c = c\cosh\frac{x}{c} - c$$

$$\mathbf{y = c\cosh\frac{x}{c}} \tag{18}$$

From theorem, we know that hyperbolic cosine square minus the hyperbolic sine square is equal to 1:

$$\cosh^2 x - \sinh^2 x$$
$$= \frac{(e^y + e^{-y})^2 - (e^y - e^{-y})^2}{2}$$
$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{2}$$
$$= 1$$

Next, to find the difference between the vertical distance y and length of cable s, the steps below are deduced base on (18) and (16):

$$y^2 - s^2$$
$$= c^2 cosh^2\left(\frac{x}{c}\right) - c^2 sinh^2\left(\frac{x}{c}\right)$$
$$= c^2\left[cosh^2\left(\frac{x}{c}\right) - sinh^2\left(\frac{x}{c}\right)\right] = c^2$$

By definition:

$$y^2 - s^2 = c^2 \tag{19}$$

Which means the square of the vertical distance (y) between origin and the attach point minus the length (s) of the cable equals to the square of the distance between origin to the lowest point of the cable (c).

Derive from (19):

$$y^2 = c^2 + s^2$$
$$\mathrm{y} = \sqrt{c^2 + s^2} \tag{20}$$

Put (20) into (10):

$$\mathrm{T} = \omega\sqrt{c^2 + s^2} = \omega\sqrt{y^2} = \omega y$$

To conclude, at anywhere on the cable in catenary shape:

$$\mathbf{T = \omega y} \tag{21}$$

Another key component in catenary is the sag of the cable which is the distance h shows in *Diagram10* and it is a cable in catenary shape with two pivots at the same height.

Where $y = \mathrm{h} + \mathrm{c} \rightarrow h = y - c$

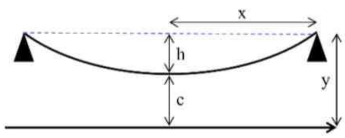


Diagram10. Catenary with same height of pivots.

Study of Structure of Bridge Design

Next, the examples of calculation of variables on catenary will be shows. However, before the first example, the equations available that calculated previously are summarized below.

Summary of Equations for Catenary Calculation

$$\boldsymbol{\omega \mathrm{c} = T_0} \quad (9)$$

$$\mathbf{T} = \boldsymbol{\omega\sqrt{c^2 + s^2}} \quad (7)$$

$$\mathbf{s = c}\ \boldsymbol{sinh\frac{x}{c}} \quad (16)$$

$$\mathbf{y} = \boldsymbol{c}\ \mathbf{cosh}\boldsymbol{\frac{x}{c}} \quad (18)$$

$$\boldsymbol{y^2 - s^2 = c^2} \quad (19)$$

$$\mathbf{T} = \boldsymbol{\omega y} \quad (21)$$

Summary of Equations for Linear Calculation

$$\mathbf{tan}\,\boldsymbol{\theta = \frac{\omega s}{F}} \quad (22)$$

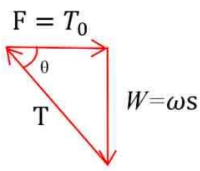


Diagram11. Triangle of forces in linear calculation.

Next, basing on the summary of equations for both catenary and linear calculations above, there will be three cases that study the relationship between variables in the structure of bridges.

1. Study of Catenary

a) Investigation of the relationship between x and T_0 & T_{max}

Known Quantity		Variables	Finding
h	20 m	x	T_0
ω	50 N/m		T_{max}
y	20 m + c		The relationship between **x** and T_0 & T_{max}
L	100 m		

Table2. Values of known quantity and finding quantity in Example1.

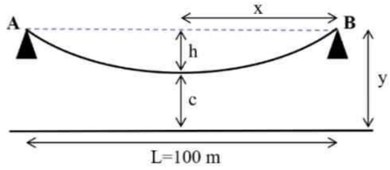


Diagram12. Catenary for Example1.

To investigate the relationship between **x** and T_0 & T_{max}, the value of x will be the variable in the following calculus and the value of T_0 and T_{max} that correspond with each x will be calculated.

First, suppose that $\mathrm{x} = 50$ m. To solve the two forces that exert on the cable, replace the value into equation (18):

$$\mathrm{y} = c \cosh \frac{x}{c}$$

$$\mathrm{h} + \mathrm{c} = c \cosh \frac{x}{c}$$

$$20 + \mathrm{c} = c \cosh \frac{50}{c}$$

Divide both sides of the equation by **c**:

$$\frac{20}{c} + 1 = \cosh \frac{50}{c}$$

Base on the equation above, the correct value for constant **c** can be derived when the left-hand side equals to the right-hand side of the equation which is the numerical method. Therefore, *Table3* is created to find the **c**. The first column of the table is variable c, where the values are settled, the second column is the right-hand side of the equation and the left column is the left-hand side of it and the values in these two columns will change as the value of **c** change.

The value of **c** increases down the table since there is still difference between the value of left-hand side and right-hand side of the equation, and the difference between each two values decrease. According to the data, when **c** is equal to 65.6, the value of both sides of the equation is similar and this means that the 65.6 is the correct value of **c**.

c	$\cosh\frac{50}{c}$	$\frac{20}{c}+1$
20	6.1323	2
40	1.8884	1.5
60	1.3678	1.3333
65	1.3107	1.3077
65.5	1.3058	1.3053
65.6	1.3048	1.3049

Table3. Solving for suspension height with numerical method.

$$\therefore c \approx 65.6 \text{ m}$$

According to the analysis diagram, the length of the cable s can be calculated:

$$y = c + h$$
$$= 65.6 + 20 = 85.6 \text{ m}$$

Also, two tensions on labeled on the cable can be calculated out base on (9) & (7).

$$T_0 = \omega c$$
$$= 50 \times 65.6 = 3280\ N$$
$$\text{T} = \omega y$$
$$= 50 \times 85.6 = 4280\ N$$

Therefore, the length of the cable **s**, the tension towards the higher pivot $\boldsymbol{T_0}$ and the tension on the lowest point of the cable **T** can be calculated out.

x (m)	$\boldsymbol{T_0}$ (N)	T (N)
50	3280	4280
60	4665	5665
70	6250	7250
80	8165	9165
90	10000	11000
100	12500	13500

Table4. Values of independent variable $\boldsymbol{T_0}$ **&** **T** and dependent variable **x.**

With the calculation for x=50 shows above, the tensions on the cable for other x values can also be deduced numerically as the *Table4* shows. Also, a *Diagram13* is made base on the table below.

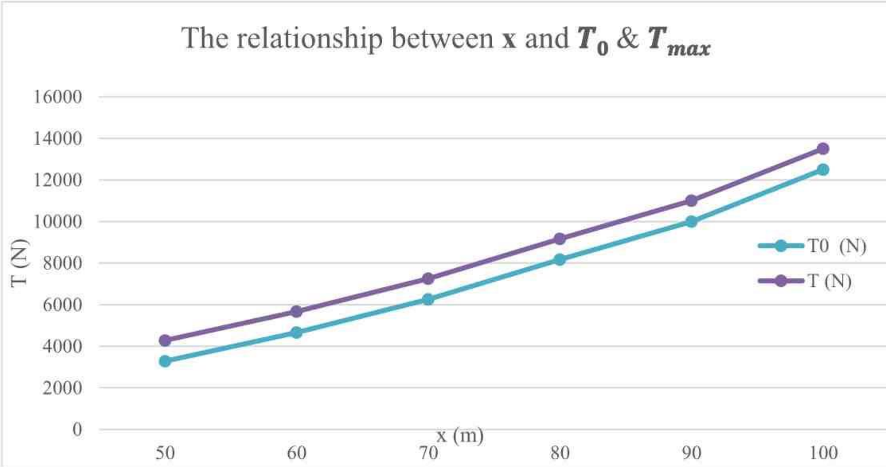


Diagram13. Relationship between **x and** T_0 & T_{max}.

According to the *Diagram13* that shows the trend of T_0 **&** **T** relative to the increase of the value of **x**, the relationship between **x** and T_0 and the relationship between **x** and **T** is proportional. As the value of x increases, the tensions exert on the cable will be increased.

b) Investigation of the relationship between ω and x in triangular bridge

Known Quantity		Variable	Finding
θ	60°	$\frac{m}{l}$	**x**
F	500 N		The relationship between **ω** and **x**
h	100 m		
s	150 m		

Table5. Values of known quantity and finding quantity.

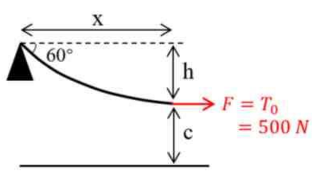


Diagram14. Catenary for Example2.

To investigate the relationship between **ω** and **x**, the value of $\frac{m}{l}$ will be the variable in the following calculus and the value of **x** that correspond with each $\frac{m}{l}$ will be calculated.

First, suppose that $\frac{m}{l} = 5$ kg.

$$\frac{m}{l} = 5\ kg$$

$$\therefore \omega = \frac{mg}{l} = (5\text{ kg})(9.8\text{ m } sec^{-2})$$

$$\omega = 49\text{ N/m}$$

According to (9):

$$c = \frac{T_0}{\omega}$$

$$= \frac{500\ N}{49\text{ N/m}} = 10.2\ m$$

Base on (19), **y** can be calculated.

$$y = \sqrt{s^2 + c^2}$$

$$= \sqrt{150^2 + 10.2^2}$$

$$= 150.3\text{ m}$$

According to (18),

$$\frac{y}{c} = \cosh\frac{x}{c}$$

$$\frac{x}{c} = \cosh^{-1}\left(\frac{y}{c}\right)$$

$$x = c\cosh^{-1}\left(\frac{y}{c}\right)$$

$$= (10.2)\cosh^{-1}\left(\frac{150.3}{10.2}\right)$$

$$= 34.51\text{ m}$$

Therefore, the distance between the highest and lowest point of the cable **h**, the distance horizontal distance of the cable **x** and the length of the cable **s** can be calculated out.

With the process above, the distance **x** for other $\frac{m}{l}$ values can also be deduced as the following *Table6* shows.

$\frac{m}{l}$ (kg)	ω (N/m)	x (m)
5	49	34.51173
10	98	20.78791
15	147	15.2372
20	196	12.16164
25	245	10.18465
30	294	8.797256

Table6. Values of independent variable $\frac{m}{l}$ and dependent variable **ω** and **x.**

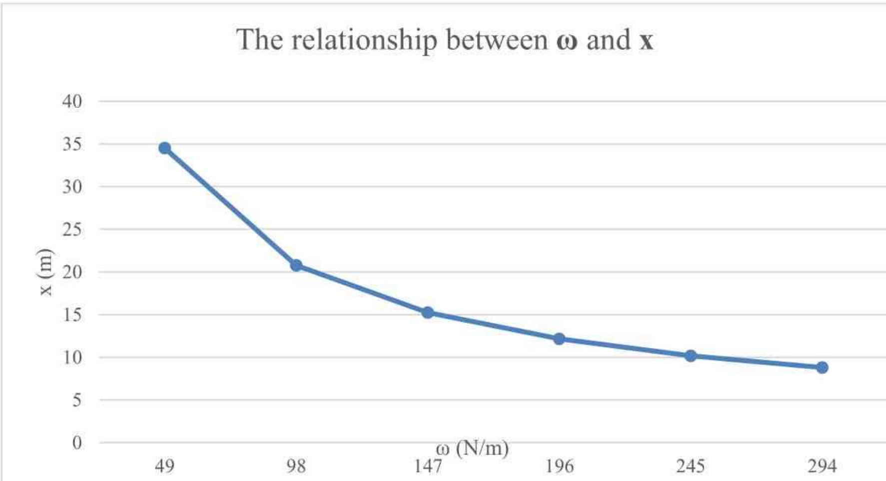


Diagram15. Relationship between ω and **x.**

According to the *Diagram15* that shows the trend of x relative to the increase of the value of ω, the relationship between ω and **x** is inverse proportional. As the value of ω increases, the distance x will decrease.

2. Study of Linear Bridge

a) investigation of the relationship between ω and x

Known Quantity		Finding
$\frac{m}{l}$	5 kg	**The relationship between θ and F**
ω	49 N/m	
s	17.67 m	

Table7. Values of known quantity and finding quantity for 2(a).

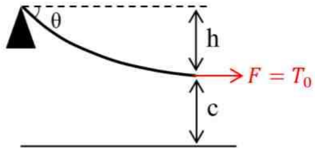


Diagram16. Catenary for 2(a).

Base on trigonometry and equation (22),

$$\tan\theta = \frac{\omega s}{F}$$

$$\theta = \tan^{-1}\frac{\omega s}{F}$$

$$= \tan^{-1}\left(\frac{(49)(17.67)}{F}\right)$$

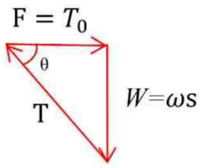


Diagram17. The triangle of forces on cable in Example3.

Therefore, the relationship between **θ** and **F** can be formulated by the equation above. The following *Table8* shows the change of the angle **θ** as the value of force **F** increases.

F (N)	θ (Degree)
0	90
200	77
400	65
600	55
800	47
1000	41

Table8. Values of independent variable **θ** and dependent variable **F** for 2(a).

Thus, according to the data shows in *Table8*, one conclusion can be found. As the force on the cable increase, the angle between the horizontal line and the cable, $\boldsymbol{\theta}$ decreases as *Diagram18* shows.

$$\theta_4 > \theta_3 > \theta_2 > \theta_1$$

$$F1 < F2 < F3 < F4$$

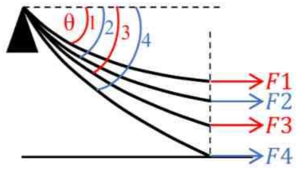


Diagram18. The change of angle between horizontal line and cable.

b) Analysis of Typical Cable-stayed Bridge Design

In the real design of the cabled stayed bridges, the structure and material of the bridges should be determined after considering the natural environment, traffic situation, length of the bridge and some other factors. Most importantly, the structure design must have the capability to support the weight of the bridge itself and the traffics pass through it. Following, a few types of cable-stayed bridge will be introduced and analyzed based on the force analysis of catenary.

First, *Diagram19* shows the commonest type of cable-stayed bridge which has support pillars and cables attach from the top of the pillar to the deck of the bridge. In this case, the distance between each connect point of the cable on the deck is evenly distributed. Deducing from the calculus in 2(a), smaller the angle of between the catenary cable and the horizontal line pass through the top point of the pillar, higher the tension at the end of the cable on deck. Which means, tension at the end of cable 4 is the largest. From cable 4 to 1, the angle increases and the tension decrease.

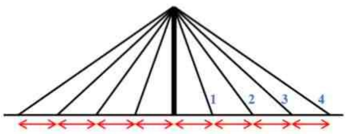


Diagram19. Real structure1 of Cable-Stayed bridge.

Most of this type of bridge are constructed to connect banks with long distribution distance, therefore, two or more same supporting pillars are required for one bridge and *Diagram20* shows a structure with two pillars. Since the tension on the furthest cable (from the pillar) is

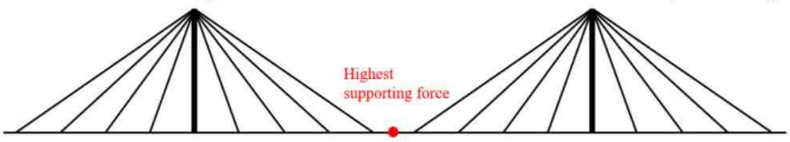


Diagram20. Weakest supporting point on the structure1 of cable-stayed bridge.

the strongest, the cars drive passing the middle point of the bridge has the highest supporting force.

However, higher tension on cable means the cable is delicate that easier to break. Therefore, some designer will design the bridge as *Diagram21* shows, where the cable connected on the first and the second pillar are cross connected to decrease the weight that each cable at the center has to carried. Although this design can support the bridge deck, it is a waste of materials.

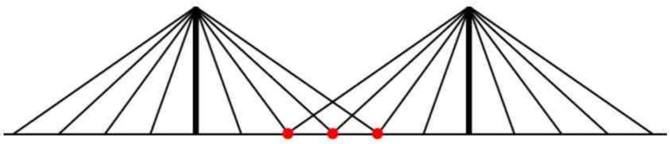


Diagram21. Weakest supporting point on the structure1 of cable-stayed bridge.

Another type of cable stayed bridge which is wildly used in modern society is the bridge with parallel cables as *Diagram22* shows. This type of bridge has the advantage of uniform distribution of forces on the bridge, since all the angles between the cable and the horizontal direction have the same value that the force for each cable to carry is the same. Therefore, this type of bridge is more stable and safer than the first type. Also, during the construction, it will safe a little more material than the first type of design.

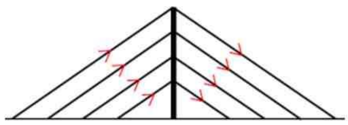


Diagram22. Real structure2 of Cable-Stayed bridge.

Moreover, for some bridges that constructed for small distance transportation, the structure of it is not symmetrical that the distance between junctions of cables on both sides of the pillar are different. This style is the rearranged base on structure1 of the Cable-Stayed bridge in *Diagram19*. The distance between cables on the left-side of the pillar in *Diagram23* is less than the right-side cable, since the foundation of the pillar in this structure is built near the bank that

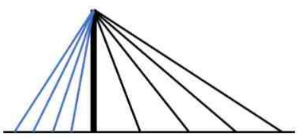


Diagram23. Real structure3 of Cable-Stayed bridge.

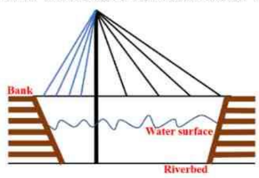


Diagram24. Structure3 of Cable-Stayed bridge in real situation.

the distance on both sides are not equal as *Diagram24* shows. The pillar of the bridge is offsite from the center of the river. This design is benefit for construction of the bridge that its easier to build since at the center of the river, the riverbed is deeper and filled with sludge that will increase the degree of difficulty for construction. Also, denser the cables on one side means less cable is used here. Therefore, it also saves the materials.

To summarize the features of different types of cable-stayed bridge design and evaluate their advantages and disadvantages, *Table9* is made to show details. In this summary, all four types have the same condition that the height of the pillar and all the materials are identical.

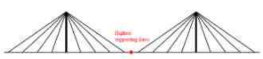
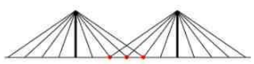
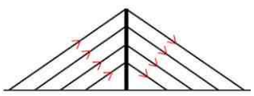
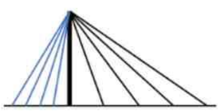
Type of Cable-Stayed Bridge	Features	Advantages	Disadvantages
	Cables are extended from the top point of the pillar. Evenly distributed junctions on deck.	Symmetrical and artistic.	Cables at the center of the bridge are easy to break since it has.
	Cables are extended from the top point of the pillar. Evenly distributed junctions on deck. Cross-connected junctions at the center.	More stable since the force for each cable to carry is lower.	Waste of materials.
	Pallela cables. uniform distribution of forces on the bridge.	More stable, safer and saving a little more material	Required to make multi junctions on the pillar
	Not symmetrical that the distance between junctions of cables on both sides of the pillar are different.	Convenient to construct.	Only suitable for short distance bridge.

Table9. Feathers and evaluation of different type of cable-stayed bridge.

To conclude that all different types of Cable-Stayed bridge have mixed features that they all have advantages and disadvantages. However, these evaluations only base on this particular type of bridge, Cable-Stayed bridge. All other types of bridge, such as arch bridge and Steel Frame Bridge are not considered as cross reference.

Conclusion

Catenary is the shape of the idealized cable or chain that hangs under its own weight when there are two supporting point at its two ends. In this case of study, the variables in the catenary structure is analyzed and summarized into a few equations. Also, the analysis of the linear situation of the cable bridge is also explained. According to the equations that summarized, some investigations of the relationship are made to explore the feature of the bridge cables.

First, the relationship between distance $\mathbf{x}$ and $\boldsymbol{T_0}$ and the relationship between $\mathbf{x}$ and $\mathbf{T}$ are studied. The result indicates that for the cable in bridge with a fixed height h, weight per unit length ω and distance between two supporting points, the longer the distance between two pivots, the greater the tensions that exert on the cable. Which means, as there is one end of the cable is connected to the top point of the bridge pillar, the further of the other end connected to the bottom of the pillar, the higher the tension exert on the cable. This refers that the cable far from the pillar is easy to break and for the bridge have two pillars, the middle part of the bridge is the most delicate.

Next, the relationship between $\boldsymbol{\omega}$ and $\mathbf{x}$ is investigated under the condition of the fix height of the pillar h, the length of the cable s, the angle between T and horizontal axis and the force on horizontal axis. The result shows that if the value of ω increases, the distance x will decrease, which means with a higher the ω, smaller the distance of the cable can provide the same supporting force for the bridge. This result indicate that the material of the bridge is essential for the construction since it can alternate the maximum stress that the bridge can support.

Except the catenary structure of the bridge, the linear structure is also studied that the relationship between the force on the cable and the angle between the horizontal line and the cable $\boldsymbol{\theta}$. The result indicates that the relationship between them is inversely proportional that as the force increase. Which means the further the connected point on the deck, the higher the tension on the cable. According to this result, some analysis of typical cable-bridge design is made, and their advantages and disadvantages are evaluated.

However, there are still some lack of consideration exist. For example, the bridge is constructed above the water in the open area without any blocks. Therefore, the wind will produce pressure on the pillar, deck and the cables. In this case, the wind power is not considered in the force analysis on the cable. Also, as the investigation shows, different material has different features, refer in particular to weight per unit length in this case. And this suggests that different cable-stayed bridges have cables in varies material. Moreover, the costs of different material are unequal. In the actual design and construction of the bridge, the engineer should not only consider the structure feathers, but also the budget.

Although the study of cases of bridge is separated into two parts, one with catenary structure and the other one with linear structure. There are still some points that can be improved in both situations. For the study of catenary structure, the equations are only suitable for the structure that has a certain suspension height. If there is no suspension height like the actual bridge design that one point connects with the top of the pillar and the other one with the deck, the equations is not applicable. However, for the actual cable-stayed bridge cable, there is still catenary exist. However, the study of linear bridge just ignores it. To improve these two limitations, the analysis of the cable-stayed bridge can combine these two unconsidered things to do more exploration and achieve a more comprehensive analysis of catenary.

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