IB Subject(s): Mathematics

Extended Essay

Line Stitching and Area:

How to find the area underneath a polygon when two adjacent sides of a square are divided into n equal parts and the corresponding points on two sides are connected

Word Count: <4000

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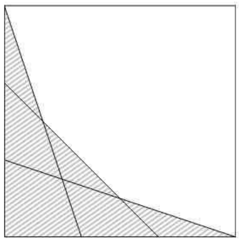
Introduction

Research question: How to measure the area underneath the line segments obtained through connecting points of $n(n \in Z^{+})$ equal sections accordingly from two adjacent sides of a unit square?

In the 2016 Senior Team Mathematics Challenge National Final organized by United Kingdom Mathematics Trust, I came across a problem (attached on the next page) that I was not able to solve within the time limit. Thus, I want to explore the problem with multiple methods in the form of an Extended Essay and try to find extensions of the problem.

QUESTION 8

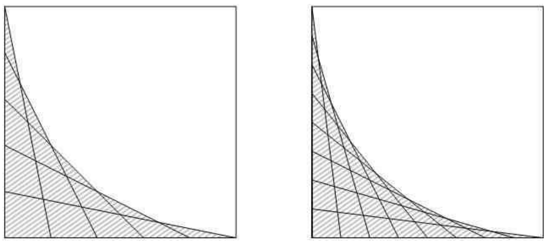
(a)



Two adjacent sides of a unit square are trisected and lines drawn between the points as shown.

What fraction of the square is hatched? [3 marks]

(b)



Two adjacent sides of a unit square are divided into n equal parts and a similar pattern is formed. (The resulting areas for $n = 5$ and $n = 8$ are shown above.)

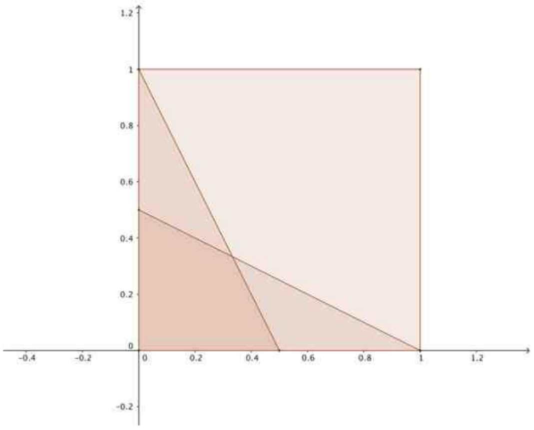
For which value of n is the hatched area equal to one-fifth of the area of the square? [3 marks]

1

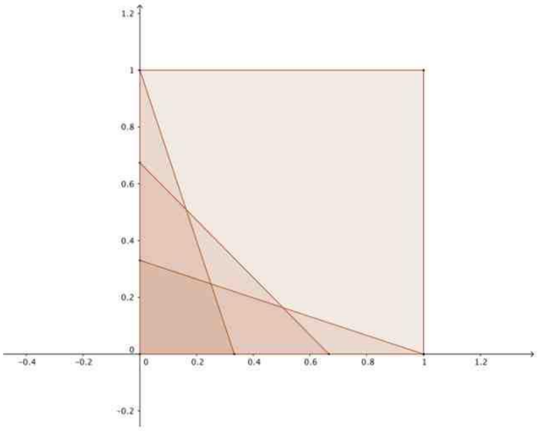
[1] "Senior Maths Team Challenge 2016-17 National Final Group Round" Accessed on 22 June 2017, http://furthermaths.org.uk/docs/GroupFinal1617.pdf

Graphs

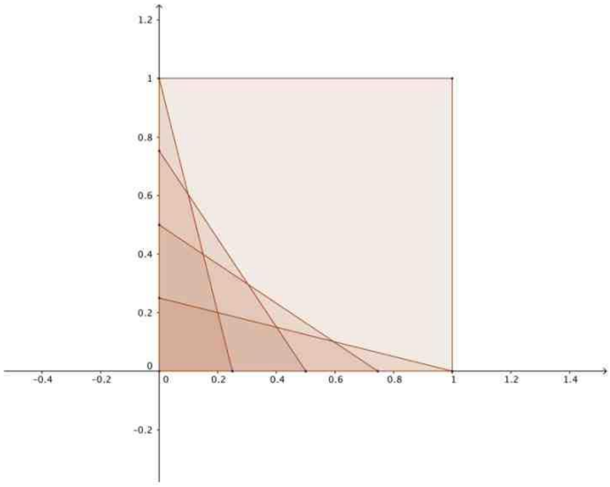
The graphs are drawn for a range of n values to illustrate the effect of the operation described in the problem. They are set into a Cartesian coordinate system for the sake of exploration.



$n = 2$



$n = 3$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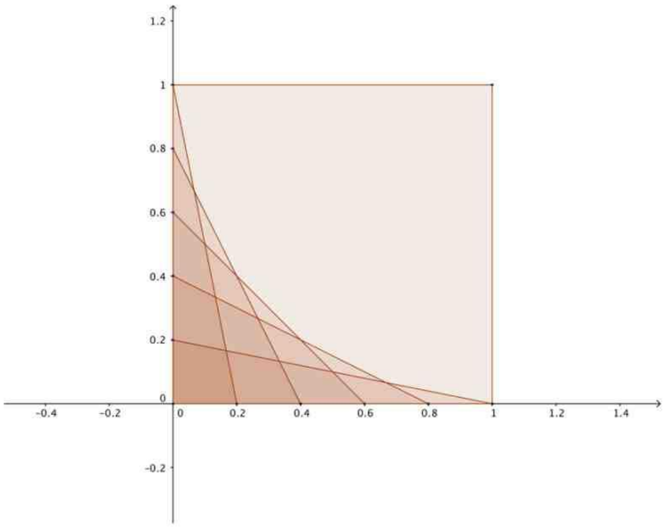
$n = 4$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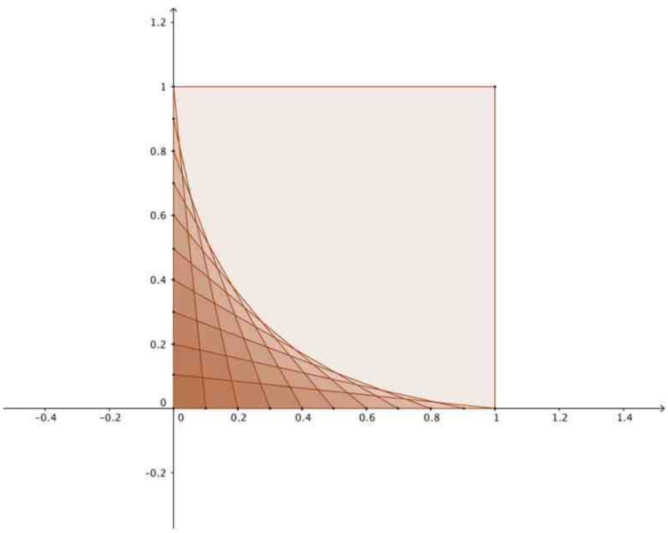
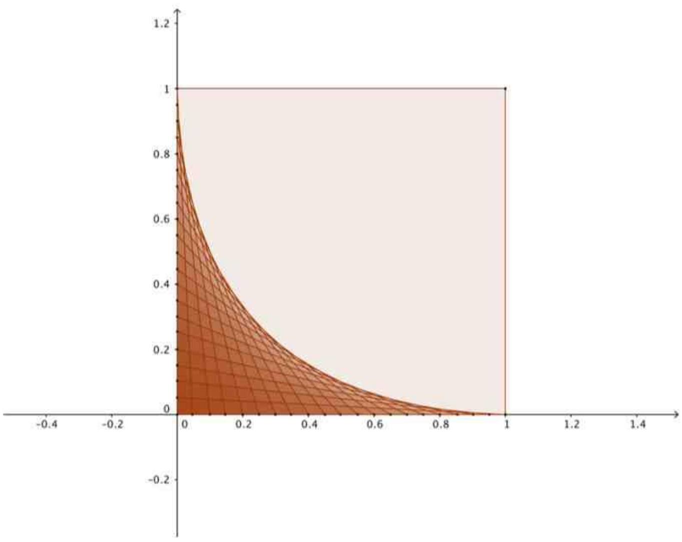
$n = 5$



$n = 10$



$n = 20$

The triangle division and grouping method

Let the x coordinate of the points on the x axis be $\frac{i}{n}(i \in Z^{+})$ and let the y coordinate of the points on the y axis be $\frac{j}{n}(j \in Z^{+})$ so that the first point from the original on the x axis in the graph $n = 20$ will have the coordinates of $\left(\frac{1}{20}, 0\right)$, which is joined to point $\left(0, \frac{20}{20}\right)$.

Connect the points accordingly i.e. join $\left(\frac{i}{n}, 0\right)$ and $\left(0, \frac{j}{n}\right)$ given that $i + j = n + 1$.

Therefore, the points can be represented as $\left(\frac{i}{n}, 0\right)$ and $\left(0, \frac{n+1-i}{n}\right)$.

It is necessary to obtain the coordinates of all the intersection points on the line segments above the hatched area since the purpose of the method is to calculate the hatched area by adding up all the triangles (details of grouping will be explained later). In order to find the coordinates of the intersection points, the general equations of of the joint line going through the two points obtained above is required.

To find the equation , the respective slope of different equations is $\frac{y_1-y_2}{x_1-x_2} = \frac{\frac{n+1-i}{n}-0}{0-\frac{i}{n}} =$

$\frac{i-1-n}{i} (i \in Z^+)$.

The general equation of the segments is $y = \frac{i-1-n}{i}x + \frac{n+1-i}{n}$.

To make a distinction between general equations and specific equations used below, the unknown number will be represented as $k (k \in Z^+)$ instead of i.

For $i = k$, the equation of the line connecting $\left(\frac{k}{n}, 0\right)$ and $\left(0, \frac{n+1-k}{n}\right)$ is $y_1 = \frac{k-1-n}{k}x +$

$\frac{n+1-k}{n}$.

The adjacent set of points of $\left(\frac{k}{n}, 0\right)$ and $\left(0, \frac{n+1-k}{n}\right)$ is $\left(\frac{k+1}{n}, 0\right)$ and $\left(0, \frac{n-k}{n}\right)$.

Therefore, the equation of the line connecting the second set of points is $y_2 = \frac{k-n}{k+1}x +$

$\frac{n-k}{n}$

To find the intersection of the two lines:

$$\begin{cases} y = \dfrac{k-1-n}{k}x + \dfrac{n+1-k}{n} & ① \\ y = \dfrac{k-n}{k+1}x + \dfrac{n-k}{n} & ② \end{cases}$$

$$\therefore \frac{k-1-n}{k}x + \frac{n+1-k}{n} = \frac{k-n}{k+1}x + \frac{n-k}{n}$$

$$\left(\frac{k-1-n}{k} - \frac{k-n}{k+1}\right)x = \frac{n-k}{n} - \frac{n+1-k}{n}$$

$$\frac{(k-1-n)\times(k+1) - k\times(k-n)}{k\times(k+1)}x = -\frac{1}{n}$$

$$\frac{k^2 + k - k - 1 - nk - n - k^2 + nk}{k\times(k+1)}x = -\frac{1}{n}$$

$$\frac{-n-1}{k\times(k+1)}x = -\frac{1}{n}$$

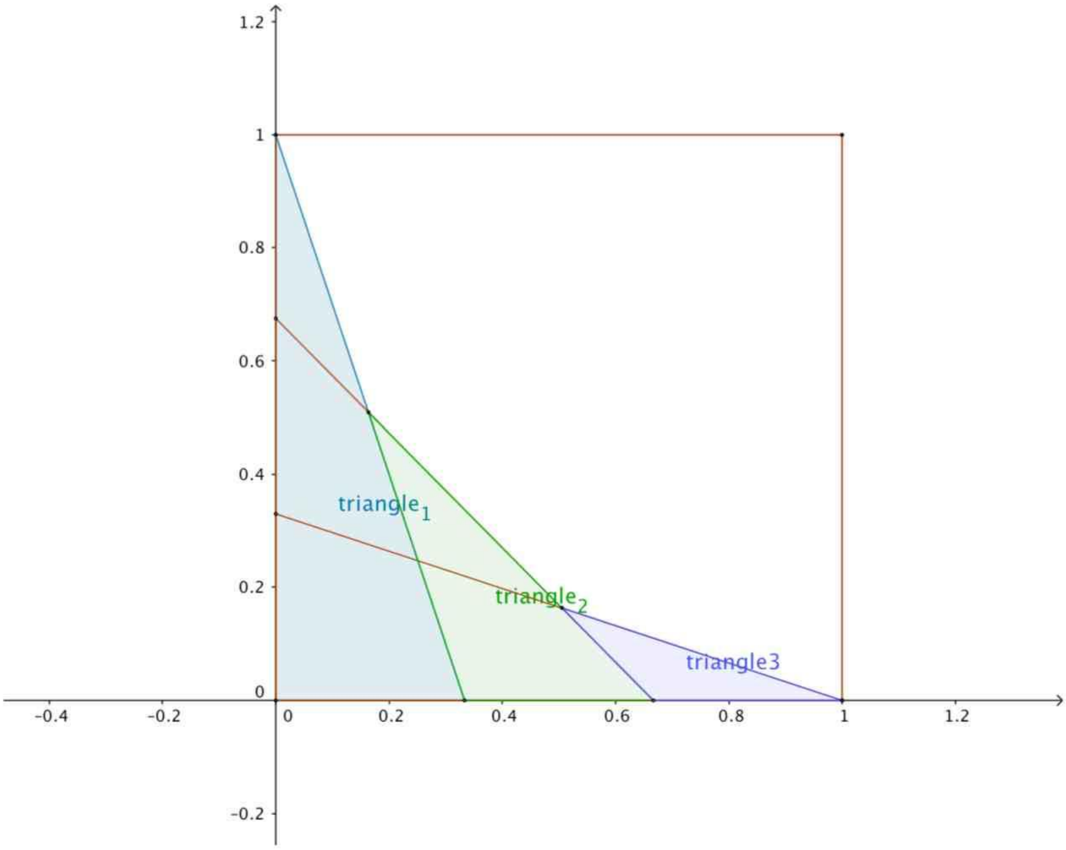
$$x = \frac{k\times(k+1)}{n\times(n+1)}$$

Substitute the value of x into function ② to find the value of y

$$y = \frac{k-n}{k+1}x + \frac{n-k}{n} = \frac{k-n}{k+1}\times\frac{i\times(i+1)}{n\times(n+1)} + \frac{n-k}{n} = \frac{k\times(k-n)}{n\times(n+1)} + \frac{n-k}{n}$$

$$= \frac{k\times(k-n) + (n-k)\times(n+1)}{n\times(n+1)} = \frac{(n-k)\times(n-k+1)}{n\times(n+1)}$$

$$\therefore \begin{cases} x = \dfrac{k\times(k+1)}{n\times(n+1)} \\ y = \dfrac{(n-k)\times(n-k+1)}{n\times(n+1)} \end{cases}$$



Take the graph of $n = 3$ as an example: in order to find out the area of hatched area, the hatched area is divided into three different triangles labelled triangle 1, triangle 2, and triangle 3($n = 3$) as the graph above highlights, which share a common base and have the height as the y coordinate of the corresponding intersecting points. The area can be calculated by the sum of the three triangles which requires the sum of all ys. Similarly, the hatched area of any n can be calculated by dividing the area into n different triangle.

To find out the sum of all ys with respect to all the ks:

$$\sum_{k=1}^{n-1} \frac{k^2 - (2n+1)\times k + n\times(n+1)}{n\times(n+1)} = \frac{1}{n\times(n+1)}\sum_{k=1}^{n-1} k^2 - \frac{(2n+1)}{n\times(n+1)}\sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1}\frac{n\times(n+1)}{n\times(n+1)}$$

$$= \frac{1}{n\times(n+1)}\sum_{k=1}^{n-1} k^2 - \frac{(2n+1)}{n\times(n+1)}\sum_{k=1}^{n-1} k + \sum_{k=1}^{n-1} 1$$

$$= \frac{1}{n\times(n+1)}\times\frac{(n-1)\times n\times(2n-1)}{6} - \frac{(2n+1)}{n\times(n+1)}\times\frac{(1+n-1)\times(n-1)}{2}$$

$$+ (n-1) = \frac{(n-1)\times(2n-1)}{6\times(n+1)} - \frac{(2n+1)\times(n-1)}{2\times(n+1)} + (n-1)$$

$$= \frac{(n-1)\times(2n-1) - 3\times(2n+1)\times(n-1) + 6\times(n-1)\times(n+1)}{6\times(n+1)}$$

$$= \frac{2n^2 - 3n + 1 - 6n^2 + 3n + 3 + 6n^2 - 6}{6\times(n+1)} = \frac{2n^2 - 2}{6\times(n+1)} = \frac{2(n-1)(n+1)}{6\times(n+1)}$$

$$= \frac{n-1}{3}$$

Include the special case of the points' set $\left(\frac{1}{n}, 0\right)$ and $(0,1)$.

Sum of all y

$$y = \frac{n-1}{3} + 1 = \frac{n+2}{3}$$

Sum of the area of all triangles

$$Area = \frac{1}{2}\times\frac{1}{n}\times\frac{n+2}{3} = \frac{n+2}{6n}$$

Therefore, the answers to the problems are evident with this formula.

For (a), $n = 3$. Thus, the hatched area is $\frac{3+2}{6\times 3} = \frac{5}{18}$ and the hatched fraction of the square is $\frac{\frac{5}{18}}{1\times 1} = \frac{5}{18}$.

For (b),

$$\frac{\frac{n+2}{6n}}{1\times1}=\frac{1}{5}$$

$$\frac{n+2}{6n}=\frac{1}{5}$$

$$5(n+2)=6n$$

$$5n+10=6n$$

$$n=10$$

To find the limiting area of the polygon under this operation:

n is a positive natural number so n is non-zero.

$$\lim_{n\to\infty}\frac{n+2}{6n}=\lim_{n\to\infty}\frac{1+\frac{2}{n}}{6}=\frac{1+0}{6}=\frac{1}{6}$$

The integration method applied to the limiting case

The limiting case of the problem will be explored from a different angle here.

The details of obtaining basic correlation of y coordinates and x coordinates have been presented in the previous method and will not repeated here.

To find out the relationship between the y coordinates and x coordinates of these intersecting points:

Make k into an equation of x,

$$x = \frac{k\times(k+1)}{n\times(n+1)}$$

$$x = \frac{k^2+k}{n\times(n+1)}$$

$$k^2 + k - (n^2+n)x = 0$$

Using the quadratic formula to find the expression for k,

$$k = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1+4(n^2+n)x}}{2}$$

k is a positive natural number so $k = \frac{-1+\sqrt{1+4(n^2+n)x}}{2}$.

Substitute $k = \frac{-1+\sqrt{1+4(n^2+n)x}}{2}$ into the expression for y.

$$y = \frac{(n-k)\times(n-k-1)}{n\times(n+1)} = \frac{k^2-(2n+1)k}{n\times(n+1)} + 1$$

$$= \frac{\left(\frac{-1+\sqrt{1+4(n^2+n)x}}{2}\right)^2 - (2n+1)\left(\frac{-1+\sqrt{1+4(n^2+n)x}}{2}\right)}{n\times(n+1)} + 1$$

$$= \frac{\frac{1-2\sqrt{1+4(n^2+n)x}+1+4(n^2+n)x}{4} + \frac{(2n+1)-(2n+1)\sqrt{1+4(n^2+n)x}}{2}}{n\times(n+1)} + 1$$

$$= \frac{\frac{2(n^2+n)x+(2n+2)-(2n+2)\sqrt{1+4(n^2+n)x}}{2}}{n\times(n+1)} + 1$$

$$= \frac{(n^2+n)x+(n+1)-(n+1)\sqrt{1+4(n^2+n)x}}{n\times(n+1)} + 1$$

n is a positive natural number so $n+1$ is non-zero.

$$y = \frac{nx+1-\sqrt{1+4(n^2+n)x}}{n} + 1$$

n is a positive natural number so n is non-zero.

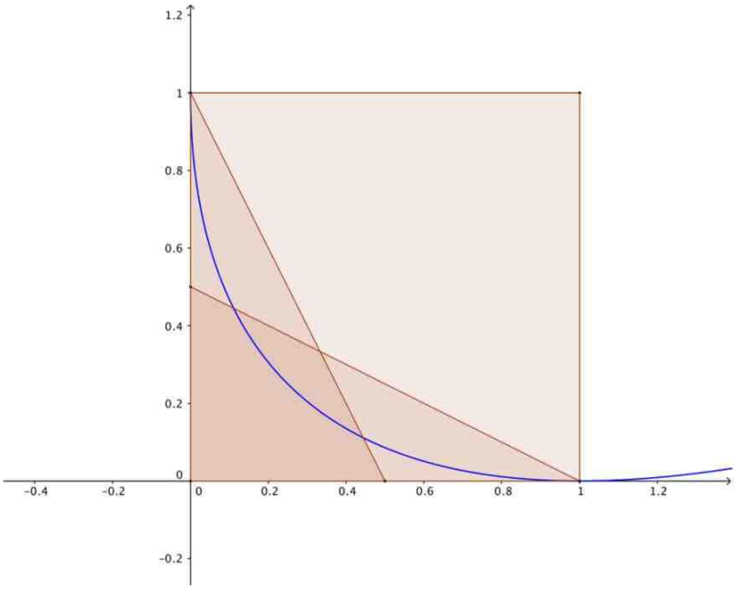
$$y = \frac{x+\frac{1}{n}-\sqrt{\frac{1}{n^2}+4(1+\frac{1}{n})x}}{1} + 1$$

When n approaches infinity, the boundary/broken line will tend to be a smooth curve, whose equation is:

$$\lim_{n\to\infty} \frac{x+\frac{1}{n}-\sqrt{\frac{1}{n^2}+4(1+\frac{1}{n})x}}{1} + 1 = \frac{x+0-\sqrt{0+4(1+0)x}}{1} + 1 = 1-2\sqrt{x}+x$$

$$\therefore y = x-2\sqrt{x}+1 = (1-\sqrt{x})^2$$

The function derived above is superimposed to the previous graphs.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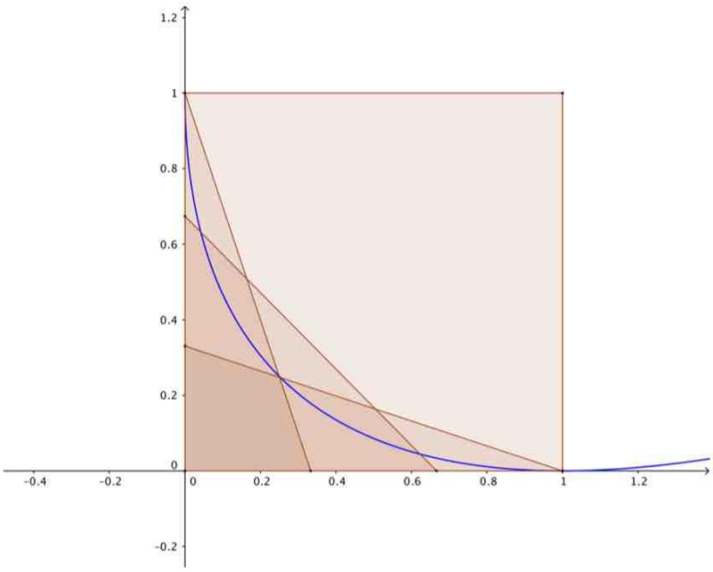
$n = 2$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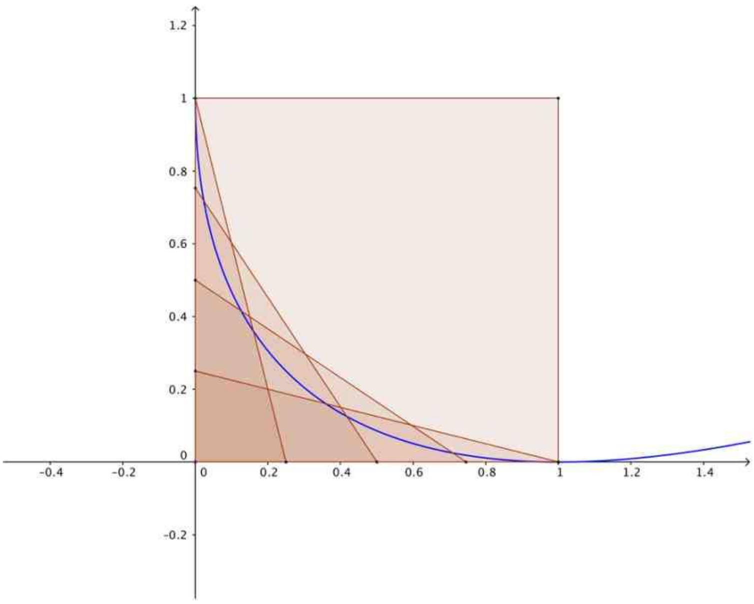
$n = 3$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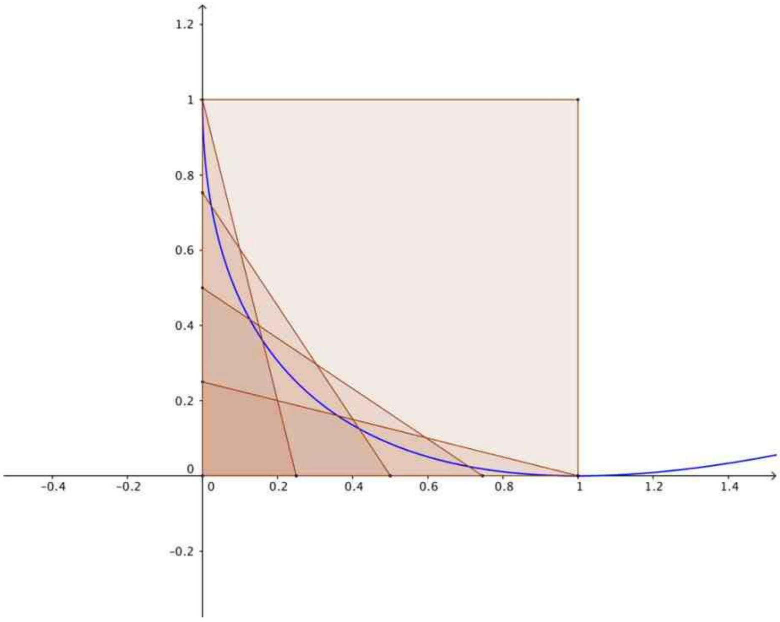
$n = 4$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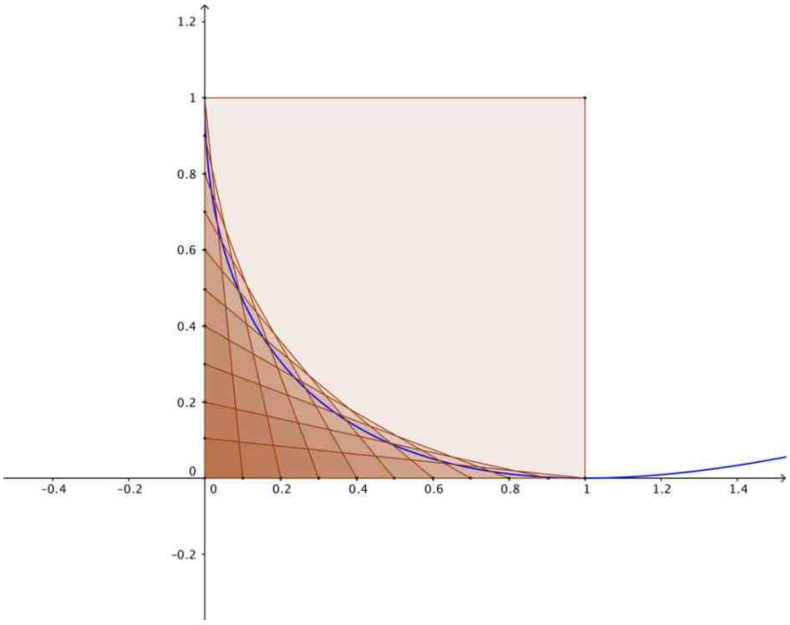
$n = 5$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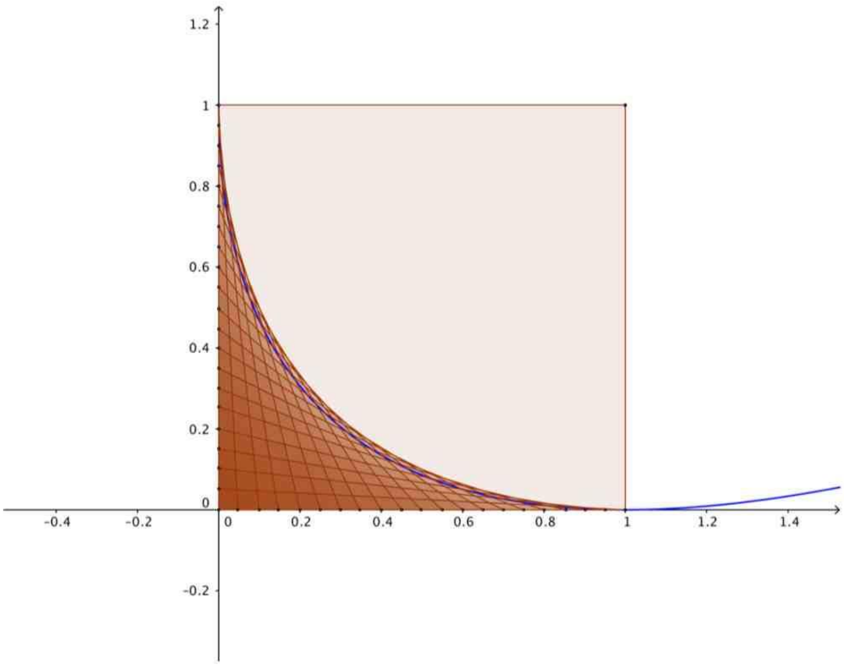
$n = 10$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$n = 20$

As the graphs show, the area underneath the curve between 0 and 1 gets overlapped with the shaded area. It is clear on the graphs that the curve is always below the line segments. Thus, the area underneath it will always be lower than the shaded area in any particular case. Pleasingly, as pattern of the graphs show, the curves fit better with the polygon as n increases. Also, the area underneath it will keep approaching the area of the shaded area.

$$\int_0^1 (x - 2\sqrt{x} + 1)\,dx = \left[\frac{1}{2}x^2 - \frac{4}{3}x^{\frac{3}{2}} + x\right]_0^1 = \left[\frac{1}{2}\times 1^2 - \frac{4}{3}\times 1^{\frac{3}{2}} + 1\right] - \left[\frac{1}{2}\times 0^2 - \frac{4}{3}\times 0^{\frac{3}{2}} + 0\right]$$

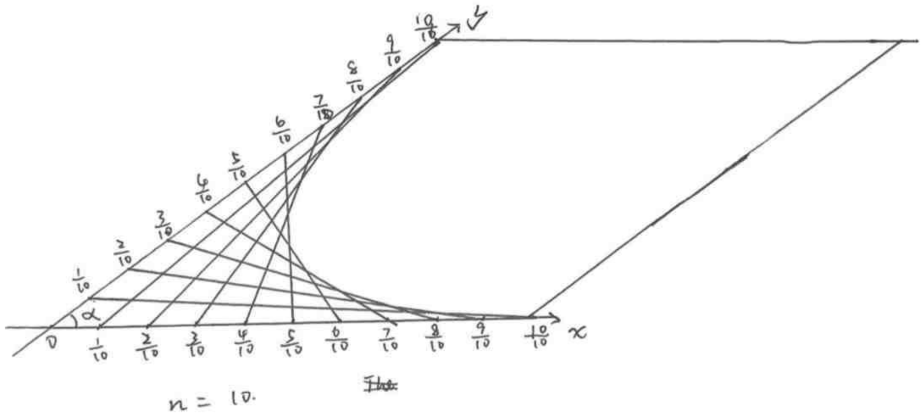
$$= \frac{1}{6} - 0 = \frac{1}{6}$$

Hence, the area underneath the curve matches the area of all the shaded area when n tends to infinity. Therefore, the second method is appropriate for the problem, giving a correct solution.

Extension

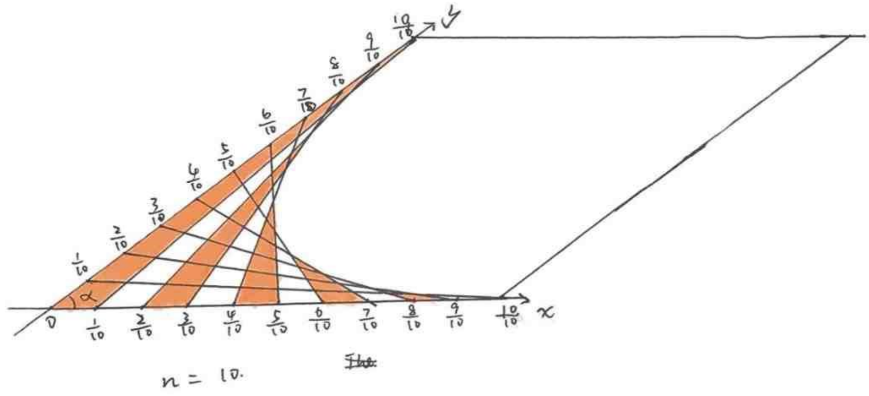
In this section, the problem will be altered a little bit and be explored in the new conditions after the change.

Change the angle between axis



First, I would start with having the two axis be at angle of α (different from 90 degrees). As the graph above shows, the square in the original question is compressed to a rhombus with interior angles of α and $\pi - \alpha$.

Note that all the fractions are left in the original form without simplification for the sake of clarity ($n = 10$).



As the graph above shows, the triangles are separated and grouped in the same way that the main section used. The orange colouring is added so that the borders between neighbouring triangles are clear. All the coordinates for any points on the graph is the same as that in the main section. Therefore, the results will be used again in this part but the proves will not be repeated.

The coordinates for the intersection points on the line segments above the hatched area is still $\begin{cases} x = \frac{k\times(k+1)}{n\times(n+1)} \\ y = \frac{(n-k)\times(n-k+1)}{n\times(n+1)} \end{cases}$. However, the y coordinate of the point in this case is not the height of its responsive triangle. Instead, the height will the $\sin\alpha\, y$.

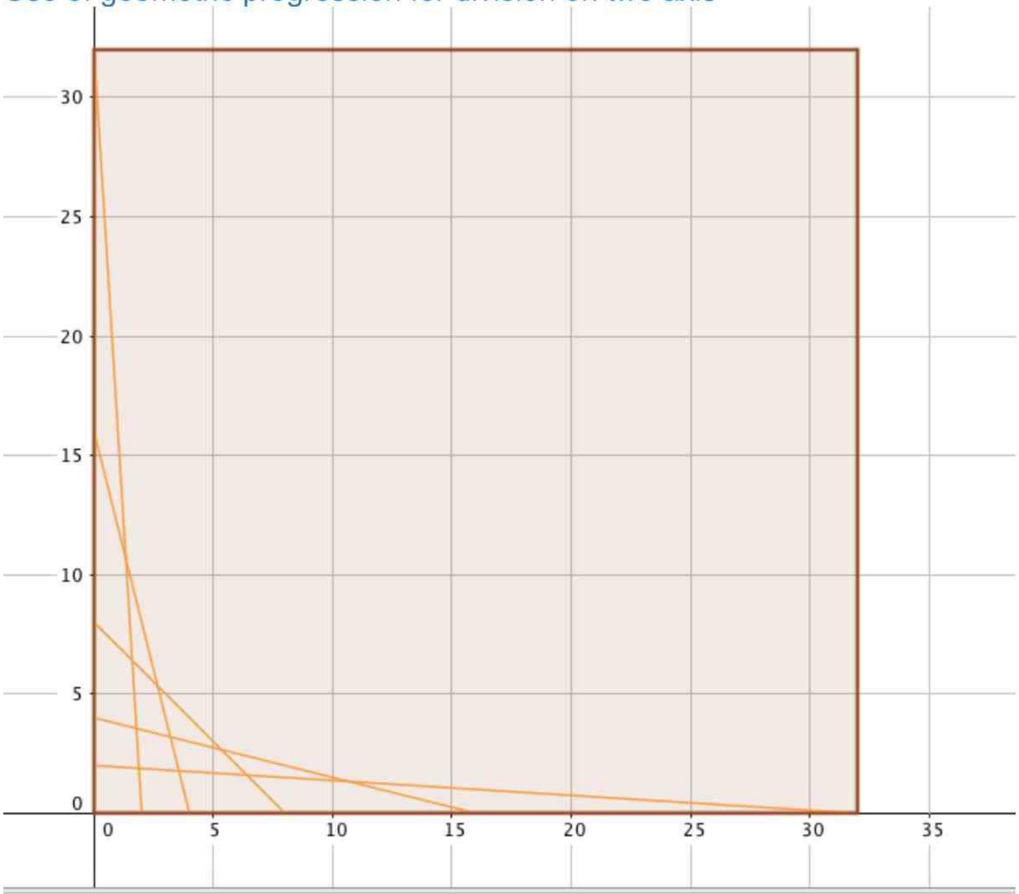
Thus, in this case, the y coordinate has an angle of $\frac{\pi}{2} - \alpha$ with the vertical. The vertical distance from the x axis will be $\cos\frac{\pi}{2} - \alpha\, y = \sin\alpha\, y$.

The multiplication of $\sin\alpha$ on the height of the triangle has carry-on effect on the rest of the expression including the hatched area.

Since the sum of all the y coordinates in the previous section is $y = \frac{n+2}{3}$, the sum of all the y coordinates(i.e. sum of heights) here is $\frac{(n+2)\sin\alpha}{3}$.

Therefore, the expression for the area of the individual triangles will be $\frac{1}{2} \times height \times base = \frac{1}{2}\sin\alpha\, y\frac{1}{n} = \frac{\sin\alpha y}{2n}$. The expression for the whole area will be $\frac{(n+2)\sin\alpha}{6n}$.

Use of geometric progression for division on two axis



For the sake of clarity of the graph, the range of the x and y coordinates is no longer restricted to $[0,1]$.

As the graph above shows, x coordinates of the points on the x axis and y coordinates of the points on the y axis follow are all terms of the same geometric sequence in order respectively.

Let the first term of the geometric sequence be a_1 and the common ratio be q. Thus, the ith term of the sequence a_i is a_1q^{i-1}.

With the similar method before, the expression for the line between $(a_k, 0)$ and $(0, a_{n+1-k})$ is $y = -\frac{a_{n+1-k}}{a_k}x + a_{n+1-k}$. The expression of the line between $(a_{k+1}, 0)$ and $(0, a_{n-k})$ is $y = -\frac{a_{n-k}}{a_{k+1}}x + a_{n-k}$. To find out the y coordinate of the point of intersection of the lines, the two equations are altered to be an equation for y.

$$\begin{cases} x = \dfrac{a_k}{a_{n+1-k}}(a_{n+1-k} - y) \\ x = \dfrac{a_{k+1}}{a_{n-k}}(a_{n-k} - y) \end{cases}$$

$$\frac{a_k}{a_{n+1-k}}(a_{n+1-k} - y) = \frac{a_{k+1}}{a_{n-k}}(a_{n-k} - y)$$

$$a_k - \frac{a_k}{a_{n+1-k}}y = a_{k+1} - \frac{a_{k+1}}{a_{n-k}}y$$

$$\left(\frac{a_{k+1}}{a_{n-k}} - \frac{a_k}{a_{n+1-k}}\right)y = a_{k+1} - a_k$$

$$\frac{a_{k+1}a_{n+1-k} - a_k a_{n-k}}{a_{n-k}a_{n+1-k}}y = a_{k+1} - a_k$$

$$\therefore y = \frac{(a_{k+1} - a_k)a_{n-k}a_{n+1-k}}{a_{k+1}a_{n+1-k} - a_k a_{n-k}} = \frac{(a_1q^k - a_1q^{k-1})a_1q^{n-k-1}a_1q^{n-k}}{a_1q^k a_1q^{n-k} - a_1q^{k-1}a_1q^{n-k-1}} = \frac{a_1q^{n-k}(q-1)}{q^2 - 1}$$

$$= \frac{a_1q^{n-k}(q-1)}{(q+1)(q-1)} = \frac{a_1q^{n-k}}{q+1}$$

Therefore, the area of the triangle with the three vertexes of the intersection point above, $(a_k, 0)$, and $(a_{k+1}, 0)$ will be $\frac{1}{2}\times\frac{a_1q^{n-k}}{q+1}\times(a_{k+1} - a_k) = \frac{1}{2}\times\frac{a_1q^{n-k}}{q+1}\times a_1q^{k-1}(q-1) =$

$\frac{{a_1}^2 q^{n-1}(q-1)}{2(q+1)}$. Since the final expression of area does not include the variable k, the area of the triangles are only determined by a_1, q and n, which means all the triangles have the same area.

The sum of all areas will be $\frac{1}{2}a_1 a_n + \frac{{a_1}^2 q^{n-1}(q-1)n}{2(q+1)} = \frac{{a_1}^2 q^{n-1}}{2}(1 + \frac{(q-1)n}{(q+1)})$.

Conclusion

This essay explores the ways to find the area of a polygon from two different perspectives. Both of them put the original shape in the coordinate system for the ease of calculating the distance between certain points. The first one relies on the splitting and grouping small triangles into big one and used basic formula for the area of triangles. The second one turns to the method of integration on the limiting case and calculated the area when n tends to infinity. For extension, the conditions of the questions are altered so that it is less restricting. In the former extension, the shape of square is replaced by rhombus and the area after the change was explored. In the latter extension, the sides are marked according to geometric progression as opposed to the arithmetic progression in the first part.

Bibliography

1. "Senior Maths Team Challenge 2016-17 National Final Group Round." Accessed on 22 June 2017. http://furthermaths.org.uk/docs/GroupFinal1617.pdf